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The Fucik Spectrum of General Sturm–Liouville Problems

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Consider the boundary value problem

$$-(pu')' + qu = \alpha u^+ - \beta u^-, \qquad \text{in} \quad (0, \pi),$$

$$c_{00} u(0) + c_{01} u'(0) = 0, \qquad c_{10} u(\pi) + c_{11} u'(\pi) = 0,$$

where $u^\pm = \max\{\pm u, 0\}$. The set of points $(\alpha, \beta) \in \mathbb{R}^2$ for which this problem has a non-trivial solution is called the Fucik spectrum. When $p \equiv 1$, $q \equiv 0$, and either Dirichlet or periodic boundary conditions are imposed, the Fucik spectrum is known explicitly and consists of a countable collection of curves, with certain geometric properties. In this paper we show that similar properties hold for the general problem above, and also for a further generalization of the Fucik spectrum. We also discuss some spectral type properties of a positively homogeneous, "half-linear" problem and use these results to consider the solvability of a nonlinear problem with jumping nonlinearities.

1. INTRODUCTION

We consider the boundary value problem

$$-(pu')' + qu = \alpha u^+ - \beta u^-, \qquad \text{in} \quad (0, \pi), \tag{1.1}$$

$$c_{00} u(0) + c_{01} u'(0) = 0, \tag{1.2}$$

$$c_{10} u(\pi) + c_{11} u'(\pi) = 0, \tag{1.3}$$

where $p \in C^1([0, \pi])$, $q \in C^0([0, \pi])$, with $p > 0$ (i.e., $p(x) > 0$, $x \in [0, \pi]$), $u^\pm(x) = \max\{\pm u(x), 0\}$, $c_{i0} + c_{i1} > 0$, $i = 0, 1$, and $(\alpha, \beta) \in \mathbb{R}^2$. Let H be the set of functions $u \in H^2(0, \pi)$ satisfying the boundary conditions (1.2), (1.3), where $H^2(0, \pi)$ denotes the usual Sobolev space of order 2 on $(0, \pi)$ (here and below, all function spaces will be real), and we regard H as a Hilbert space with the $H^2(0, \pi)$ inner product. Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the usual $L^2(0, \pi)$ norm and inner product and let $S_H = \{u \in H : \|u\| = 1\}$. The set S_H

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is a smooth submanifold of H. We define the Sturm–Liouville operator $L\colon H \to L^2(0, \pi)$, by

$$Lu = -(pu')' + qu, \qquad u \in H.$$

Now, let

$$\tilde{\Sigma}(L) = \{(\alpha, \beta, u) \in \mathbb{R}^2 \times S_H : Lu = \alpha u^+ - \beta u^-\},$$

and let $\Sigma(L)$ be the projection of $\tilde{\Sigma}(L)$ onto $\mathbb{R}^2$. The set $\Sigma(L)$ is called the *Fucik spectrum* of L. This set is of importance in the study of semilinear boundary value problems with jumping nonlinearities; see, for example, [5, 9].

When $p \equiv 1$, $q \equiv 0$, and either Dirichlet or periodic boundary conditions are imposed, the Fucik spectrum of L is known explicitly, see Section 6 of [5] or [9], and consists of a countable collection of curves, with certain geometric properties (we describe these properties more fully below). Much less is known about the structure of the spectrum for more general Sturm–Liouville operators L. The Fucik spectrum of elliptic analogues of L has also been considered, but even less is known about the structure of the spectrum in this case; see, for example, [5, 6, 11, 13]. In particular, in [11] Pistoia considered the operator $A = -\Delta$ (Δ denotes the Laplace operator on a domain $\Omega \subset \mathbb{R}^n$, with Dirichlet boundary conditions) and obtained properties of $\Sigma(-\Delta)$ by studying the set $\tilde{\Sigma}(-\Delta)$.

In this paper we obtain various geometric properties of the Fucik spectrum of the general Sturm–Liouville operator L. In particular, we show that the spectrum is a collection of curves, and we show that the asymptotic behaviour of these curves is determined by the eigenvalues of certain operators which are similar to L, but involving various combinations of the boundary conditions (1.2), (1.3) and Dirichlet boundary conditions. We also consider a generalized Fucik spectrum, and show that this has similar geometric properties.

Similar methods also enable us to discuss some spectral type properties of a positively homogeneous, "half-linear" problem. We then use these results to give necessary and sufficient conditions for the solvability of a nonlinear problem, with jumping nonlinearity, in terms of the location of the "half-eigenvalues" of the associated "half-linear" problem.

2. GEOMETRIC PROPERTIES OF $\Sigma(L)$

We begin with some preliminary terminology and properties of $\tilde{\Sigma}(L)$ and $\Sigma(L)$. Let $u \in H$ be a non-zero function whose zeros in $[0, \pi]$ are simple (the derivative $u' \neq 0$ at each zero). The zeros of u in $(0, \pi)$ will be called *nodes*; the open intervals between nodes (or between a node and an end point 0 or π) will be called *nodal intervals*; a nodal interval is *positive* (respectively, *negative*) if $u > 0$ (respectively, $u < 0$) on that interval. From now on, ν will denote an element of $\{\pm\}$, i.e., either $\nu = +$ or $\nu = -$. For each $k \geqslant 1$, $\nu \in \{\pm\}$, let $S_{k,\nu}$ denote the set of functions $u \in S_H$ having only simple zeros and exactly k nodal intervals, and with $\nu u > 0$ in a deleted neighbourhood of $x = 0$ (with the obvious interpretation of νu). If $u \in S_H$ is a solution of (1.1) (or any homogeneous differential equation below) then u has only simple zeros, so $u \in S_{k,\nu}$ for some k and ν. Now let $\lambda_k = \lambda_k(L)$, $\psi_k = \psi_k(L) \in S_H$, $k = 1, 2, \ldots$, denote the eigenvalues (in increasing order) and the corresponding eigenfunctions of L. It is well known that, for each k, the eigenfunction ψ_k has exactly k nodal intervals, so the sign of ψ_k may be chosen such that $\psi_k \in S_{k,+}$.

For each $k \geqslant 1$, it is clear that the points $(\lambda_k, \lambda_k, \pm\psi_k) \in \tilde{\Sigma}(L)$, and since λ_k is a simple eigenvalue no other point of the form $(\lambda_k, \lambda_k, u)$, $u \in S_H$, can belong to $\tilde{\Sigma}(L)$. Thus, $(\lambda_k, \lambda_k) \in \Sigma(L)$, for each $k \geqslant 1$. Conversely, if $(\alpha, \alpha) \in \Sigma(L)$ then it is clear that α must be an eigenvalue λ_k. Also, since $\psi_1 \in S_{1,+}$, we have $\psi_1 > 0$ in the whole interval $(0, \pi)$, so we see that the lines $\tilde{\Gamma}_{1,+} = \{(\lambda_1, \beta, \psi_1) : \beta \in \mathbb{R}\}$, $\tilde{\Gamma}_{1,-} = \{(\alpha, \lambda_1, -\psi_1) : \alpha \in \mathbb{R}\}$, lie in $\tilde{\Sigma}(L)$, and hence the lines $\Gamma_{1,+} = \{(\lambda_1, \beta) : \beta \in \mathbb{R}\}$, $\Gamma_{1,-} = \{(\alpha, \lambda_1) : \alpha \in \mathbb{R}\}$, lie in $\Sigma(L)$ and intersect transversly at the point (λ_1, λ_1). The proof of part (i) of Theorem 3 in [5] shows that all other points $(\alpha, \beta, u) \in \tilde{\Sigma}(L)$ must have $\alpha > \lambda_1$ and $\beta > \lambda_1$. Thus, writing $\tilde{\Gamma}_1 = \tilde{\Gamma}_{1,+} \cup \tilde{\Gamma}_{1,-}$, $\Gamma_1 = \Gamma_{1,+} \cup \Gamma_{1,-}$, and defining the sets $\tilde{\Sigma}_1(L) := \tilde{\Sigma}(L) \backslash \tilde{\Gamma}_1$, $\Sigma_1(L) := \Sigma(L) \backslash \Gamma_1$, we see that the set $\Sigma_1(L)$ must lie in the quadrant $(\lambda_1, \infty)^2 \subset \mathbb{R}^2$. We will now describe some geometrical properties of the set $\Sigma_1(L)$.

THEOREM 2.1. *The set $\Sigma_1(L)$ consists of a collection of C^1 curves $\Gamma_{k,\nu}$, $k \geqslant 2$, $\nu \in \{\pm\}$, which, for each k and ν, have the properties*:

(i) *there exists a number $b_{k,\nu} \geqslant \lambda_1$ and a strictly decreasing function $\alpha_{k,\nu} \in C^1(b_{k,\nu}, \infty)$, with $\alpha_{k,\nu}(\beta) > \lambda_1$, for $\beta \in (b_{k,\nu}, \infty)$, and $\lim_{\beta \searrow b_{k,\nu}} \alpha_{k,\nu}(\beta) = \infty$, such that the curve $\Gamma_{k,\nu} = \{(\alpha_{k,\nu}(\beta), \beta) : \beta \in (b_{k,\nu}, \infty)\}$*;

(ii) *the pair of curves $\Gamma_{k,+}$, $\Gamma_{k,-}$ intersect at the point (λ_k, λ_k).*

(iii) *writing $\Gamma_k = \Gamma_{k,+} \cup \Gamma_{k,-}$, for $k \geqslant 2$, we have $\Gamma_k \cap \Gamma_{k'} = \varnothing$ for any $k, k' \geqslant 2$ with $k \neq k'$.*

Proof. For any $(\alpha, \beta, u) \in \tilde{\Sigma}(L)$, we have, by definition,

$$(L - (\alpha\chi_{\{u>0\}} + \beta\chi_{\{u<0\}}))\, u = 0, \tag{2.1}$$

where $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{x \in (0, \pi): u(x) > 0\}$; similarly for $\chi_{\{u<0\}}$. Thus, since this is an ordinary differential equation problem, we have

$$\dim \ker(L - (\alpha\chi_{\{u>0\}} + \beta\chi_{\{u<0\}})) = 1. \tag{2.2}$$

We note that this "non-degeneracy" condition is similar to condition (3.1) of [11], which is the main condition used in [11]. However, in the partial differential equation setting of [11] this condition need not hold in general, whereas in the present ordinary differential equation setting it holds automatically for all $(\alpha, \beta, u) \in \tilde{\Sigma}(L)$.

Define the function $G: \mathbb{R}^2 \times H \to \mathbb{R} \times L^2(0, \pi)$ by

$$G(\alpha, \beta, u) = (\|u\|^2, Lu - (\alpha u^+ - \beta u^-)).$$

Clearly, $\tilde{\Sigma}(L)$ is the set of solutions of the equation

$$G(\alpha, \beta, u) = (1, 0); \tag{2.3}$$

furthermore, G is C^1 near any point $\zeta_0 = (\alpha_0, \beta_0, u_0) \in \tilde{\Sigma}(L)$, and the partial Fréchet derivative of the second component of G at ζ_0 is the operator

$$u \to (L - (\alpha_0\chi_{\{u_0>0\}} + \beta_0\chi_{\{u_0<0\}}))\, u$$

(the proof uses the simplicity of the zeros of u_0 in $[0, \pi]$—details of a similar proof are given in parts (ii) and (iii) of the proof of Lemma 3.1 in [12]). Hence, we can apply the implicit function theorem to the equation (2.3) at the point ζ_0—condition (2.2) ensures that the non-singularity condition in the implicit function theorem holds at ζ_0 (the argument is similar to that in the proof of Lemma 3.3 in [11], although the function G used here is slightly different to the function G used in [11]). It follows that, in a neighbourhood of ζ_0 in $\mathbb{R}^2 \times S_H$, the set $\tilde{\Sigma}(L)$ consists of a curve of the form

$$\tilde{\Gamma}(\zeta_0) = \{(\alpha(\beta; \zeta_0), \beta, u(\beta; \zeta_0)): \beta \in (\beta_0 - \varepsilon(\zeta_0), \beta_0 + \varepsilon(\zeta_0))\},$$

where $\varepsilon(\zeta_0) > 0$, the function $(\alpha(\cdot; \zeta_0), u(\cdot; \zeta_0))$ is C^1 on the interval $(\beta_0 - \varepsilon(\zeta_0), \beta_0 + \varepsilon(\zeta_0))$, and $(\alpha(\beta_0; \zeta_0), \beta_0, u(\beta_0; \zeta_0)) = \zeta_0$. Clearly, the projection of $\tilde{\Gamma}(\zeta_0)$ onto $\mathbb{R}^2$ is the curve

$$\Gamma(\zeta_0) = \{(\alpha(\beta; \zeta_0), \beta): \beta \in (\beta_0 - \varepsilon(\zeta_0), \beta_0 + \varepsilon(\zeta_0))\} \subset \Sigma(L).$$

For now, let $\tilde{\Gamma}_{\mathrm{m}}(\zeta_0)$ denote the maximal connected component of $\tilde{\Sigma}_1(L)$ in $\mathbb{R}^2 \times S_H$ containing $\tilde{\Gamma}(\zeta_0)$. Since the above implicit function theorem construction holds at all points in $\tilde{\Gamma}_{\mathrm{m}}(\zeta_0)$, this set must be a C^1 curve of the form

$$\tilde{\Gamma}_{\mathrm{m}}(\zeta_0) = \{(\alpha_{\mathrm{m}}(\beta;\zeta_0), \beta, u_{\mathrm{m}}(\beta;\zeta_0)) \colon \beta \in (b_l(\zeta_0), b_r(\zeta_0))\},$$

with C^1 functions $\alpha_{\mathrm{m}}(\cdot\,;\zeta_0)$, $u_{\mathrm{m}}(\cdot\,;\zeta_0)$, defined on an interval $(b_l(\zeta_0), b_r(\zeta_0))$. The projection, $\Gamma_{\mathrm{m}}(\zeta_0)$, of $\tilde{\Gamma}_{\mathrm{m}}(\zeta_0)$ onto $\mathbb{R}^2$ has a similar form. Also, by the preceding remarks, $b_l(\zeta_0) \geqslant \lambda_1$ and $\alpha_{\mathrm{m}}(\cdot\,;\zeta_0) > \lambda_1$ on $(b_l(\zeta_0), b_r(\zeta_0))$.

We now discuss the geometry of the curve $\Gamma_{\mathrm{m}}(\zeta_0)$. Differentiating the second component of the equation $G(\alpha_{\mathrm{m}}(\beta), \beta, u_{\mathrm{m}}(\beta)) = (1, 0)$ with respect to β yields

$$\begin{aligned} L u'_{\mathrm{m}}(\beta) - (\alpha_{\mathrm{m}}(\beta)\, \chi_{\{u_{\mathrm{m}}(\beta) > 0\}} + \beta \chi_{\{u_{\mathrm{m}}(\beta) < 0\}})\, u'_{\mathrm{m}}(\beta) \\ - (\alpha'_{\mathrm{m}}(\beta)\, u_{\mathrm{m}}(\beta)^+ - u_{\mathrm{m}}(\beta)^-) = 0, \end{aligned}$$

for all $\beta \in (b_l, b_r)$ (here, $'$ denotes differentiation with respect to β, and we have omitted the argument ζ_0 for notational convenience). Taking the inner product of this equation with $u_{\mathrm{m}}(\beta)$ and using (2.1) yields

$$\alpha'_{\mathrm{m}}(\beta)\, \|u_{\mathrm{m}}(\beta)^+\|^2 + \|u_{\mathrm{m}}(\beta)^-\|^2 = 0.$$

Since $\alpha_{\mathrm{m}}(\beta) > \lambda_1$, $\beta > \lambda_1$, it follows from the standard spectral theory of L that $\|u_{\mathrm{m}}(\beta)^+\| \neq 0$ and $\|u_{\mathrm{m}}(\beta)^-\| \neq 0$, so we have

$$\alpha'_{\mathrm{m}}(\beta) = -\frac{\|u_{\mathrm{m}}(\beta)^-\|^2}{\|u_{\mathrm{m}}(\beta)^+\|^2} < 0. \tag{2.4}$$

Thus the function α_{m} is strictly decreasing. Now suppose that $b_r < \infty$. Since α_{m} is decreasing and bounded below (by λ_1), the one-sided limit $\alpha_r = \lim_{\beta \nearrow b_r} \alpha_{\mathrm{m}}(\beta)$ exists, with $\alpha_r \geqslant \lambda_1$. Also, by standard bootstrapping type arguments, there exists an increasing sequence $\{\beta_n\}$ such that $\lim_{n \to \infty} \beta_n = b_r$ and the limit $u_r = \lim_{n \to \infty} u_{\mathrm{m}}(\beta_n)$ exists (in H). But now, applying the above implicit function theorem argument at the point (α_r, b_r, u_r) implies that locally the solution set of (2.3) is a C^1 curve, so the curve $\tilde{\Gamma}_{\mathrm{m}}(\zeta_0)$ can be extended to the right of b_r, which contradicts the maximality of $\tilde{\Gamma}_{\mathrm{m}}(\zeta_0)$. Hence we must have $b_r = \infty$. Similarly, it can be shown that $\lim_{\beta \searrow b_l} \alpha_{\mathrm{m}}(\beta) = \infty$.

In the above constructions we started with an arbitrary point $\zeta_0 \in \tilde{\Sigma}_1(L)$. However, we have seen that, for each $k \geqslant 2$ and v, the point $(\lambda_k, \lambda_k, v\psi_k) \in \tilde{\Sigma}_1(L)$. We will denote the curve $\tilde{\Gamma}_{\mathrm{m}}(\lambda_k, \lambda_k, v\psi_k)$ in $\tilde{\Sigma}_1(L)$ by $\tilde{\Gamma}_{k,v}$, and the corresponding curve in $\Sigma_1(L)$ by $\Gamma_{k,v}$. The corresponding functions of β will be denoted by $\alpha_{k,v}$ and $u_{k,v}$, and are defined on an interval $(b_{k,v}, \infty)$. Now, the above results have shown that, for any $\zeta_0 \in \tilde{\Sigma}_1(L)$, the curve $\Gamma_{\mathrm{m}}(\zeta_0)$ through ζ_0 lies in the quadrant $(\lambda_1, \infty)^2$, is strictly decreasing, and has vertical and horizontal asymptotes. It follows that $\Gamma_{\mathrm{m}}(\zeta_0)$ must intersect the diagonal line $D=\{(s,s)\in\mathbb{R}^2 : s\in\mathbb{R}\}$ at exactly one point. In addition, our previous remarks imply that this point of intersection must be (λ_k,λ_k), for some $k\geqslant 2$, and $u_{\mathrm{m}}(\lambda_k;\zeta_0)$ must be either ψ_k or $-\psi_k$. Thus, for any initial point $\zeta_0\in\tilde{\Sigma}_1(L)$, the curve $\tilde{\Gamma}_{\mathrm{m}}(\zeta_0)$ must coincide with some curve $\tilde{\Gamma}_{k,\nu}$. This proves that the set $\tilde{\Sigma}_1(L)$ is exactly the collection of curves $\tilde{\Gamma}_{k,\nu}$, $k\geqslant 2$, $\nu\in\{\pm\}$, while the set $\Sigma_1(L)$ is exactly the corresponding collection of curves $\Gamma_{k,\nu}$.

We also note that, by continuity, $u_{k,\nu}(\beta)\in S_{k,\nu}$ for all $\beta\in(b_{k,\nu},\infty)$ (a change in the nodal properties of $u_{k,\nu}$ would imply that there exists β_0 such that $u_{k,\nu}(\beta_0)$ has a non-simple zero; since $u_{k,\nu}(\beta_0)$ satisfies a homogeneous ordinary differential equation, this implies that $u_{k,\nu}(\beta_0)=0$, but this contradicts $u_{k,\nu}(\beta_0)\in S_H$).

To prove the final result, we first prove that the result holds for $k\geqslant 2$ with $k'=k+1$. To do this we suppose the contrary, i.e., suppose that for some $k\geqslant 2$ there exists $(\alpha,\beta)\in\Gamma_k\cap\Gamma_{k+1}$, and let (α,β,u_k), (α,β,u_{k+1}) be the corresponding elements of $\tilde{\Gamma}_k$, $\tilde{\Gamma}_{k+1}$ (where $\tilde{\Gamma}_k=\tilde{\Gamma}_{k,+}\cup\tilde{\Gamma}_{k,-}$, $k\geqslant 2$). Since the number of nodal intervals of the functions u_k, u_{k+1} differs by exactly one, there is exactly one end of the interval $(0,\pi)$ at which the endmost nodal intervals of these functions have the same sign. It then follows from the uniqueness of the solution of the initial value problem for ordinary differential equations that u_k and u_{k+1} must be scalar multiples of each other in this interval, and hence, by continuation, they must be scalar multiples over the whole interval $(0,\pi)$. However, this contradicts the difference in the number of nodal intervals, so the assumption that $\Gamma_k\cap\Gamma_{k+1}\neq\varnothing$ must be incorrect.

Now, for each $k\geqslant 2$ the set Γ_k intersects the diagonal D in the point (λ_k,λ_k), and the sequence λ_k, $k=1,2,\ldots$, is strictly increasing. Thus, for each $k\geqslant 2$, it follows from the previous result and the above results on the geometry of the curves in $\Sigma(L)$ that the set Γ_k is separated from the set $\Sigma(L)\backslash(\Gamma_k\cup\Gamma_{k\pm 1})$ by the curves $\Gamma_{k\pm 1,\nu}$. Hence the result follows. This completes the proof of Theorem 2.1. ∎

3. ASYMPTOTES OF $\Sigma(L)$

Theorem 2.1 shows that the curves $\Gamma_{k,\nu}$ have vertical and horizontal asymptotes. We will now describe these asymptotes, i.e., we will obtain the value of $\alpha_\infty(k,\nu):=\lim_{\beta\to\infty}\alpha_{k,\nu}(\beta)$, and of $\beta_\infty(k,\nu):=b_{k,\nu}$. To do this we introduce some related eigenvalue problems, obtained from the eigenvalue problem for L by changing the boundary conditions. Consider the equation

$$-(pv')'+qv=\lambda v,\qquad \text{in}\quad (0,\pi),\tag{3.1}$$

and the Dirichlet boundary conditions

$$u(0) = 0, \tag{3.2}$$

$$u(\pi) = 0. \tag{3.3}$$

We define the following sets of eigenvalues $\{\lambda_k^{OO}\} = \{\lambda_k^{OO}(L)\}$, etc., associated with Eq. 3.1 and the indicated boundary conditions, in the usual manner:

$$\begin{aligned} &\{\lambda_k^{OO}\} \leftrightarrow (1.2), (1.3), \qquad &\{\lambda_k^{OD}\} \leftrightarrow (1.2), (3.3), \\ &\{\lambda_k^{DD}\} \leftrightarrow (3.2), (3.3), \qquad &\{\lambda_k^{DO}\} \leftrightarrow (3.2), (1.3), \end{aligned} \tag{3.4}$$

In this notation a superscript O denotes one of the original boundary conditions (1.2) or (1.3), and D denotes one of the Dirichlet boundary conditions (3.2) or (3.3), while the position of O or D indicates whether the condition holds at the left or right end of the interval $[0, \pi]$.

We will also require some further notation. For each $k \geqslant 1$, v, let $n_P(k, v)$ (respectively, $n_N(k, v)$) denote the number of positive (respectively, negative) nodal intervals of $v\psi_k$. These numbers satisfy $n_P(k, v) + n_N(k, v) = k$ and

$$\text{if } k \text{ is even, } n_P(k, v) = k/2;$$

$$\text{if } k \text{ is odd, } n_P(k, v) = \begin{cases} (k+1)/2, & \text{if} \quad v = +, \\ (k-1)/2, & \text{if} \quad v = -. \end{cases}$$

The vertical asymptotes are described in the following theorem.

THEOREM 3.1. *For each $k \geqslant 2$,*

$$\alpha_\infty(k, +) = \begin{cases} \lambda^{OO}_{n_P(k,\, +)}, & \textit{if } k \textit{ is odd}, \\ \lambda^{OD}_{n_P(k,\, +)}, & \textit{if } k \textit{ is even}, \end{cases}$$

$$\alpha_\infty(k, -) = \begin{cases} \lambda^{DD}_{n_P(k,\, -)}, & \textit{if } k \textit{ is odd}, \\ \lambda^{DO}_{n_P(k,\, -)}, & \textit{if } k \textit{ is even}. \end{cases}$$

Remark 3.2. Intuitively, this result can be explained as follows. Consider a fixed $k \geqslant 2$ and v. As β increases from λ_k the function $u_{k,v}(\beta) \in S_{k,v}$ starts at $u_{k,v}(\lambda_k) = v\psi_k$, and, as $\beta \to \infty$, the function $u_{k,v}(\beta)$ oscillates more and more rapidly on the negative nodal intervals, so these intervals shrink to a set of points $\{x_i\} \subset [0, \pi]$, while the positive nodal intervals converge to a set of "limiting intervals" between these points. Also, $u_{k,v}(\beta)$ converges, in $H^1(0, \pi)$, to a limiting non-negative function $u_\infty(k, v)$ which has "corners" at the points $\{x_i\}$ and is a H^2 function on the limiting intervals.

This process is illustrated in Fig. 1a–c (starting with the eigenfunction ψ_4, for a problem with Neumann boundary conditions). By changing the sign and scaling appropriately on successive limiting intervals, to make the derivative continuous, the function $u_\infty(k, \nu)$ can be transformed into a $H^2(0, \pi)$ function $\tilde{u}_\infty(k, \nu)$ which satisfies Eq. (3.1), with $\lambda = \alpha_\infty(k, \nu)$ (see Fig. 1d). The boundary conditions satisfied by $\tilde{u}_\infty(k, \nu)$ (the same as $u_\infty(k, \nu)$) are as follows: if the leftmost nodal interval of $\nu\psi_k$ is positive then $\tilde{u}_\infty(k, \nu)$ satisfies the boundary condition (1.2), while if this interval is negative then it shrinks to the point $x = 0$, and $\tilde{u}_\infty(k, \nu)$ satisfies the Dirichlet condition (3.2); a similar alternative holds at $x = \pi$. Summing up these results, we see that $\tilde{u}_\infty(k, \nu)$ is an eigenfunction of one of the eigenvalue problems (3.4), and $\alpha_\infty(k, \nu)$ is a corresponding eigenvalue—the problem, and the particular eigenvalue, is determined by the number of positive nodal intervals $n_P(k, \nu)$ and the sign of the endmost nodal intervals of the function $\nu\psi_k$. The function in Fig. 1d is the eigenfunction corresponding to λ_2^{OD}. The proof of the theorem consists of making these assertions rigorous.

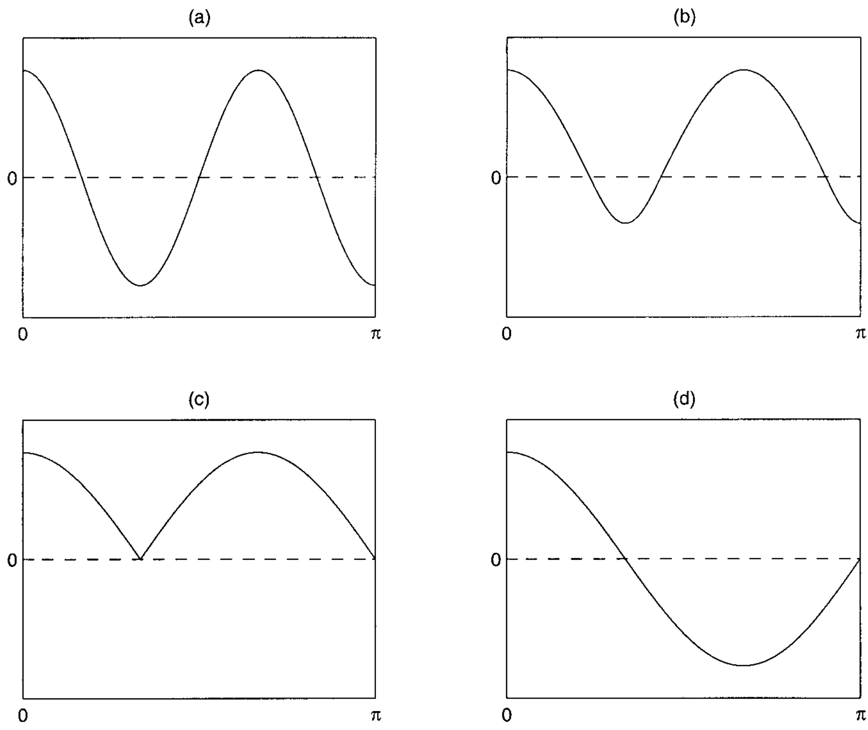

FIG. 1. Illustration of the process described in Remark 3.2. The functions shown are: (a) $u_{4,+}(\lambda_4) = \psi_4$; (b) $u_{4,+}(\beta)$, $\beta > \lambda_4$; (c) $u_\infty(4, +)$; (d) $\tilde{u}_\infty(4, +)$.

The horizontal asymptotes can be obtained in a similar manner. In this case, $\alpha \to \infty$ and the positive nodal intervals shrink to points, so the limiting values are determined by the number of negative nodal intervals and the sign of the endmost nodal intervals of the function $v\psi_k$.

COROLLARY 3.3. *For each $k \geqslant 2$,*

$$\beta_\infty(k, +) = \begin{cases} \lambda^{DD}_{n_N(k, +)}, & \text{if } k \text{ is odd}, \\ \lambda^{DO}_{n_N(k, +)}, & \text{if } k \text{ is even}, \end{cases}$$

$$\beta_\infty(k, -) = \begin{cases} \lambda^{OO}_{n_N(k, -)}, & \text{if } k \text{ is odd}, \\ \lambda^{OD}_{n_N(k, -)}, & \text{if } k \text{ is even}. \end{cases}$$

Proof of Theorem 3.1. To simplify the notation in the proof we fix k and v and omit these variables from now on; thus we write $u(\beta) = u_{k,v}(\beta)$, etc. We also write

$$p_{\max} = \max_{x \in [0, \pi]} \{p(x)\}, \qquad p_{\min} = \min_{x \in [0, \pi]} \{p(x)\}, \qquad |q|_{\max} = \max_{x \in [0, \pi]} \{|q(x)|\}.$$

In the course of the proof, C will denote a positive constant which is independent of β—its value may be different at each occurrence.

Let $N_i(\beta)$, $i = 1, ..., n_N$ (respectively, $P_i(\beta)$, $i = 1, ..., n_P$) denote the negative (respectively, positive) nodal intervals of $u(\beta)$. For any interval I, we let $|I|$ denote the length of I.

LEMMA 3.4. *There exists a constant $C > 0$ such that for $\beta > 2\,|q|_{\max}$,*

$$|P_i(\beta)| \geqslant C, \qquad i = 1, ..., n_P,$$

$$\lim_{\beta \to \infty} |N_i(\beta)| = 0, \qquad i = 1, ..., n_N.$$

Proof. For a given $\beta > 2\,|q|_{\max}$, consider a particular interval $N_i(\beta)$ and let $N_i(\beta) = (x_1, x_2)$. At least one of x_1, x_2 is a zero of $u(\beta)$, and without loss of generality we may suppose that x_1 is. Consider the initial value problem

$$p_{\max}\, v'' + (\beta - |q|_{\max})\, v = 0, \qquad v(x_1) = 0, \qquad v'(x_1) = -1, \tag{3.5}$$

and let x_3 be the first zero of the solution of (3.5) to the right of x_1 (x_3 may not be in the interval $[0, \pi]$ but this is unimportant). By the Sturm comparison theorem (Theorem 1.2, Chap. 8 of [4]), $x_2 \leqslant x_3$, that is, $|x_2 - x_1| \leqslant |x_3 - x_1|$. However, we can solve (3.5) explicitly and show that $|x_3 - x_1| \leqslant C\beta^{-1/2}$, where the constant C does not depend on x_1, x_2 or β. A similar argument proves the other result (slightly more care is required on intervals $P_i(\beta)$ with end points 0 or π—in this case we use the Prüfer angle methods described in Sections 1 and 2 of Chap. 8 of [4] to represent of the boundary condition (3.2) or (3.3) in terms of a Prüfer angle, as on p. 210 of [4], and combine this with Theorem 1.2 in [4]; Prüfer angle methods are also described in [3] for similar problems to those considered here). ∎

Now let $\|\cdot\|_1$ denote the usual norm on the Sobolev space $H^1(0, \pi)$.

LEMMA 3.5. *There exists a constant $C>0$ and a sequence $\{\beta_n\}$ such that $\lim_{n\to\infty}\beta_n=\infty$ and*

$$\lim_{n\to\infty}\|u(\beta_n)^-\|_0=0, \qquad \|u(\beta_n)\|_1\leqslant C, \quad \forall n.$$

Proof. Since $\alpha(\beta)$ is decreasing and bounded below,

$$-\int_{\lambda_k}^{\infty}\alpha'(\beta)\,d\beta<\infty,$$

so there exists a sequence $\{\beta_n\}$ such that $\lim_{n\to\infty}\beta_n=\infty$ and $|\alpha'(\beta_n)|\leqslant\beta_n^{-1}$, for all n. Hence, from (2.4) we obtain

$$\|u(\beta_n)^-\|_0^2\leqslant\beta_n^{-1},$$

which proves the first result. Now, multiplying Eq. (1.1) by $u_n=u(\beta_n)$ and integrating yields

$$\langle pu_n', u_n'\rangle=-\langle qu_n, u_n\rangle+\alpha(\beta_n)\,\|u_n^+\|_0^2+\beta_n\,\|u_n^-\|_0^2+\text{b.t.}\leqslant C(1+|\text{b.t.}|), \tag{3.6}$$

where the term b.t. arises from integration by parts and, by using the boundary conditions (1.2), (1.3), it can be shown that the Poincaré type estimate

$$|\text{b.t.}|\leqslant C(u_n(0)^2+u_n(\pi)^2)\leqslant C\,\|u_n\|_0^2+\varepsilon\,\|u_n'\|_0^2$$

holds, where $0<\varepsilon<p_{\min}/2$. This, together with (3.6), yields the second result. ∎

By Lemma 3.4 there exists a subsequence (which we will not relabel) of the sequence $\{\beta_n\}$ (from Lemma 3.5), and a set of points $x_i\in[0,\pi]$, $i=1, ..., n_N$, such that, for each i, $N_i(\beta_n)$ converges to x_i (in an obvious sense). Let P_j, $j=1, ..., n_P$, denote the maximal open intervals in the set $(0, \pi)\backslash\{x_i\}$, i.e., for each j, the interval P_j is the limit of the positive nodal interval $P_j(\beta_n)$. By Lemma 3.4 the intervals $P_j(\beta_n)$ cannot shrink to points. In particular, the points x_i are distinct and there are exactly n_P intervals P_j. By Lemma 3.5 the sequence $u(\beta_n)$ is bounded in $H^1(0, \pi)$ so there exists a subsequence of $\{\beta_n\}$ (which again we will not relabel) such that $u_n := u(\beta_n) \rightharpoonup u_\infty$ (where $\rightharpoonup$ denotes weak convergence in $H^1(0, \pi)$). Also, it follows from the compactness of the embedding $H^1(0, \pi) \to C^0(0, \pi)$ that $u_n \to u_\infty$ in $C^0(0, \pi)$, and hence it is clear that $u_\infty(x_i) = 0$ for each i.

We now consider the case where each of the endmost nodal intervals of $v\psi_k$ are negative. Fix j, $1 \leqslant j \leqslant n_P$, and let $\phi \in C_0^\infty(P_j)$. Then, multiplying (1.1) by ϕ and integrating yields (writing $\alpha_n = \alpha(\beta_n)$),

$$
\begin{aligned}
0 &= \langle pu_n', \phi' \rangle + \langle qu_n, \phi \rangle - \alpha_n \langle u_n^+, \phi \rangle + \beta_n \langle u_n^-, \phi \rangle \\
&\to \langle pu_\infty', \phi' \rangle + \langle qu_\infty, \phi \rangle - \alpha_\infty \langle u_\infty, \phi \rangle
\end{aligned}
$$

(using $u_n \rightharpoonup u_\infty$ and the fact that, for n sufficiently large, $u_n^- = 0$ and $u_n^+ = u_n$ on the support of ϕ). Hence, since $C_0^\infty(P_j)$ is dense in $H_0^1(P_j)$, we obtain

$$
\langle pu_\infty', \phi' \rangle + \langle qu_\infty, \phi \rangle - \alpha_\infty \langle u_\infty, \phi \rangle = 0, \qquad \forall \phi \in H_0^1(P_j).
$$

Thus, on each interval P_j the function $\xi_j := u_\infty|_{P_j} \in H_0^1(P_j)$ (since $u_\infty = 0$ at the end points of P_j), and ξ_j is a weak solution of the equation

$$
-(pu')' + qu - \alpha_\infty u = 0, \tag{3.7}
$$

so by standard regularity results, $\xi_j \in H^2(P_j) \cap H_0^1(P_j)$ for each $j = 1, ..., n_P$. In particular, $\xi_j \in C^1(\bar{P}_j)$. We now suppose that $\xi_j \neq 0$ for each $j = 1, ..., n_P$, (this assumption will be justified below) and we construct a function $\tilde{u}_\infty$ on $[0, \pi]$ as follows: on $\bar{P}_1$, let $\tilde{u}_\infty|_{\bar{P}_1} = \xi_1$; on $\bar{P}_2$, let $\tilde{u}_\infty|_{\bar{P}_2} = \gamma_2 \xi_2$, where $\gamma_2 \in \mathbb{R}$ is chosen so that $\tilde{u}_\infty|_{\bar{P}_1 \cup \bar{P}_2} \in C^1(\bar{P}_1 \cup \bar{P}_2)$; continuing in this manner, by enforcing continuity of the derivative $\tilde{u}_\infty'$ across the points $\{x_i\}$, we construct the function $\tilde{u}_\infty$ on $[0, \pi]$. It can be seen that: (i) $\tilde{u}_\infty \in H^2(0, \pi)$; (ii) $\tilde{u}_\infty$ satisfies (3.7) on $[0, \pi]$; (iii) $\tilde{u}_\infty$ satisfies the Dirichlet boundary conditions (3.2), (3.3). Thus, $\tilde{u}_\infty$ is an eigenfunction and α_∞ is an eigenvalue of the problem (3.1), (3.2), (3.3). Since $\tilde{u}_\infty$ has n_P nodal intervals, we have $\alpha_\infty = \lambda_{n_P}^{DD}$. This proves Theorem 3.1, in this case, once we have justified the above assumption that $\xi_j \neq 0$, $j = 1, ..., n_P$. The cases where the endmost intervals are not negative can be proved in a similar manner, using the usual weak formulation of the boundary conditions at 0 and π.

We will now show that $\xi_j \neq 0$, $j = 1, ..., n_P$, using the following lemmas.

LEMMA 3.6. *Suppose that* $\beta > 2\,|q|_{\max}$ *and* $N = N_j(\beta) = (x_1, x_2)$, *for some* j, $1 \leqslant j \leqslant n_N$. *If* $\bar{N} \subset (0, \pi)$, *then there exists a constant* $C > 0$, *independent of* β, j, *such that*

$$C^{-1} < -\frac{u'(\beta)(x_1)}{u'(\beta)(x_2)} < C.$$

Proof. Suppose that the minimum of $u = u(\beta)$ on $N(\beta)$ occurs at x_0. Multiplying Eq. (1.1) by pu' yields

$$\frac{d}{dx}(pu')^2 = -(\beta - q)\, p\, \frac{d}{dx} u^2, \qquad \text{in } N(\beta),$$

and integrating from x_0 to x_2 gives (using $u'(x_0) = u(x_2) = 0$ and $u' > 0$ on (x_0, x_2))

$$(\beta - |q|_{\max})\, p_{\min}\, u(x_0)^2 \leqslant (p(x_2)\, u'(x_2))^2 \leqslant (\beta + |q|_{\max})\, p_{\max} u(x_0)^2,$$

and hence,

$$\frac{(\beta - |q|_{\max})\, p_{\min}}{p_{\max}^2}\, u(x_0)^2 \leqslant u'(x_2)^2 \leqslant \frac{(\beta + |q|_{\max})\, p_{\max}}{p_{\min}^2}\, u(x_0)^2.$$

Similar estimates hold for $|u'(x_1)|$, and combining these results proves the lemma. ∎

LEMMA 3.7. *Suppose that* $\beta > 2\,|q|_{\max} + \lambda_k$ *and* $P = P_j(\beta)$, *for some* j, $1 \leqslant j \leqslant n_P$. *If* x_0 *is an end point of* P *with* $u(\beta)(x_0) = 0$, *then there exists a constant* $C > 0$, *independent of* β, j, *such that,*

$$C^{-1}\,|u'(\beta)(x_0)| \leqslant \|u(\beta)\|_{L^2(P)} \leqslant C\,|u'(\beta)(x_0)|.$$

Proof. By applying a variant of Gronwall's inequality (see the argument on p. 379 of [2]), it can be shown that $|u(\beta)(x)| \leqslant C\,|u'(\beta)(x_0)|$, $x \in P$, so the second inequality follows immediately. Now suppose, without loss of generality, that x_0 is the left end point of P, and write $P = (x_0, x_1)$. Integrating (1.1) over (x_0, x_1) yields

$$\begin{aligned} p(x_0)\, u'(\beta)(x_0) &\leqslant C \int_{x_0}^{x_1} u(\beta)\, dx + p(x_1)\, u'(\beta)(x_1) \\ &\leqslant C(\|u(\beta)\|_{L^2(P)} + |u(\beta)(x_1)|) \leqslant C\,\|u(\beta)\|_{L^2(P)} \end{aligned}$$

(the final inequality is obtained by using similar estimates to those in the Poincaré inequality at the end of the proof of Lemma 3.5). ∎

Now suppose that our above assumption is false and that $\xi_{j_0}=0$ for some j_0, $1 \leqslant j_0 \leqslant n_P$. Then we can choose a sequence $\{n(r): r \in \mathbb{N}\}$ such that $\lim_{r\to\infty} n(r)=\infty$, and $\|u(\beta_{n(r)})\|_{L^2(P_{j_0}(\beta_{n(r)}))} \leqslant 1/r$, for each integer $r \geqslant 1$. For each $r \geqslant 1$ we can apply Lemmas 3.6 and 3.7 to successive intervals $P_j(\beta_{n(r)})$, $N_j(\beta_{n(r)})$, to show that $\|u(\beta_{n(r)})^+\|_0 \leqslant C/r$ (where C is independent of r). But this contradicts the above results that $\|u(\beta)\|_0 \equiv 1$ (by construction) and $\|u(\beta_{n(r)})^-\|_0 \to 0$ (by Lemma 3.5), so our above assumption must be true. This completes the proof of Theorem 3.1.

4. A GENERALIZED FUCIK SPECTRUM

The standard spectrum of L is often generalized by considering equations of the form

$$Lu = \lambda a u, \tag{4.1}$$

for $a \in L^\infty(0,\pi)$ with $a>0$ almost everywhere. It is well known that there is a sequence of generalized eigenvalues of this problem, with the usual properties—we will denote these eigenvalues by $\lambda_k(L; a)$, $k \geqslant 1$. We can define a generalized Fucik spectrum of L by considering the equation

$$Lu = \alpha a u^+ - \beta b u^-, \tag{4.2}$$

where $a, b \in L^\infty(0,\pi)$ and $a, b > 0$ almost everywhere. We will use the notation $\Sigma(L; a, b)$, $\tilde{\Sigma}(L; a, b)$, for the generalized version of the sets $\Sigma(L)$, $\tilde{\Sigma}(L)$, defined previously (the definition of these sets is clear). The first obvious difference between the generalized problem and the previous one is that, if $a \neq b$, then putting $\alpha=\beta$ in (4.2) does not reduce it to the standard (generalized) eigenvalue problem (4.1). In particular, we do not immediately obtain elements of $\tilde{\Sigma}(L; a, b)$ in the set $D \times S_H$ in this way, as we did in the previous case. Thus it is not clear, at first sight, that the set $\tilde{\Sigma}(L; a, b)$ is non-empty. However, if we can show that this set is non-empty then the arguments of Sections 2 and 3, based on the implicit function theorem and continuation from the points $(\lambda_k, \lambda_k, \nu\psi_k) \in \tilde{\Sigma}(L)$, can also be applied here, in a similar manner (a slight extension of the argument in Lemma 3.1 of [12] shows that the mapping $u \to \alpha a u^+ - \beta b u^-$ is C^1 near any function $u_0 \in H$ having only simple zeros in $[0,\pi]$). Thus we first prove the following lemma.

LEMMA 4.1. *For each $k \geqslant 1$, $\nu \in \{\pm\}$, there exists a unique solution*

$$(\beta^0_{k,\nu}, \beta^0_{k,\nu}, u^0_{k,\nu}) \in D \times S_{k,\nu}$$

of (4.2) $(\beta^0_{k,\nu} = \beta^0_{k,\nu}(L; a, b),\ u^0_{k,\nu} = u^0_{k,\nu}(L; a, b))$. *If* $k' > k \geqslant 1$ *then*

$$\beta^0_{k',\nu'} > \beta^0_{k,\nu}, \qquad \text{for each} \quad \nu', \nu \in \{\pm\}. \tag{4.3}$$

Proof. We consider the auxiliary problem

$$Lv = \gamma(av^+ - bv^-) + \mu v, \tag{4.4}$$

where $\gamma, \mu \in \mathbb{R}$. When $\gamma = 0$ this is the standard eigenvalue problem for L (with spectral parameter μ), so there is a sequence of solutions $(\gamma, \mu, v) = (0, \lambda_k, \nu\psi_k) \in \mathbb{R}^2 \times S_{k,\nu}$, $k \geqslant 1$, $\nu \in \{\pm\}$. For each k, ν, the implicit function theorem argument in the proof of Theorem 2.1 can be applied to yield a maximal curve $(\gamma, \mu_{k,\nu}(\gamma), v_{k,\nu}(\gamma)) \in \mathbb{R}^2 \times S_{k,\nu}$ of solutions of (4.4), defined on some neighbourhood of $\gamma = 0$, with $(\mu_{k,\nu}(0), v_{k,\nu}(0)) = (\lambda_k, \nu\psi_k)$. As in the proof of (2.4), it can be shown that

$$\mu'_{k,\nu}(\gamma) = -\langle av_{k,\nu}(\gamma)^+, v_{k,\nu}(\gamma)^+\rangle - \langle bv_{k,\nu}(\gamma)^-, v_{k,\nu}(\gamma)^-\rangle, \tag{4.5}$$

and hence there is a constant $C > 0$ (depending only on a and b) such that

$$-C^{-1} \leqslant \mu'_{k,\nu}(\gamma) \leqslant -C.$$

It follows from this that the above curves are defined for all $\gamma \in \mathbb{R}$, and also, for each $k \geqslant 1$, $\nu \in \{\pm\}$, there exists a unique point $\gamma_{k,\nu}$ such that $\mu_{k,\nu}(\gamma_{k,\nu}) = 0$, and hence, defining

$$\beta^0_{k,\nu} := \gamma_{k,\nu}, \qquad u^0_{k,\nu} := v_{k,\nu}(\gamma_{k,\nu}),$$

it follows that $(\beta^0_{k,\nu}, \beta^0_{k,\nu}, u^0_{k,\nu}) \in D \times S_{k,\nu}$ is a solution of (4.2).

To prove uniqueness, suppose that there exists another solution $(\tilde{\beta}^0, \tilde{\beta}^0, \tilde{u}^0) \in D \times S_{k,\nu}$. Then the continuation argument can be applied, starting at this point and allowing γ to vary from $\gamma = \tilde{\beta}^0$ to $\gamma = 0$, to yield a solution $(\tilde{\mu}, \tilde{u}) \in \mathbb{R} \times S_{k,\nu}$ of the standard eigenvalue problem, different to $(\lambda_k, \nu\psi_k)$. But this contradicts the standard spectral theory of L, so the point $(\tilde{\beta}^0, \tilde{\beta}^0, \tilde{u}^0)$ cannot exist.

Finally, to prove (4.3) we choose any $\nu, \nu', k \geqslant 1$ and observe that when $\gamma = 0$ we have $\mu_{k+1,\nu'}(0) > \mu_{k,\nu}(0)$ and during the continuation process the curves in $\mathbb{R}^2$ given by $(\gamma, \mu_{k,\nu}(\gamma))$, $(\gamma, \mu_{k+1,\nu'}(\gamma))$, $\gamma \in \mathbb{R}$, cannot intersect, hence (4.3) must hold (these results follow from similar arguments to those used in the proof of part (iii) of Theorem 2.1). This completes the proof of the lemma. ∎

Remark 4.2. When $k = 1$, the function $u^0_{1,+} > 0$ on $(0, \pi)$, so by definition,

$$Lu^0_{1,+} = \beta^0_{1,+} a u^0_{1,+},$$

i.e., $\beta^0_{1,+}(L; a, b) = \lambda_1(L; a)$. Similarly, $\beta^0_{1,-}(L; a, b) = \lambda_1(L; b)$. Thus, as for the usual Fucik spectrum, the generalized Fucik spectrum $\Sigma(L; a, b)$ contains two lines, $\Gamma_{1,\pm}(L; a, b)$, parallel to the axes; in this case these lines intersect at the point $(\lambda_1(L; a), \lambda_1(L; b))$, which in general will not lie on D if $a \neq b$. Letting $\Sigma_1(L; a, b) = \Sigma(L; a, b) \backslash (\Gamma_{1,+}(L; a, b) \cup \Gamma_{1,-}(L; a, b))$, it follows from similar arguments to those in the standard case that $\Sigma_1(L; a, b)$ lies in the quadrant $(\lambda_1(L; a), \infty) \times (\lambda_1(L; b), \infty)$.

To describe the asymptotes of the set $\Sigma_1(L; a, b)$ we will require sets of generalized eigenvalues $\{\lambda_k^{DD}(L; a)\}$, etc., which are defined in terms of the equation

$$-(pv')' + qv = \lambda av, \qquad \text{in } (0, \pi),$$

and various combinations of the boundary conditions (1.2), (1.3), (3.2), (3.3), in the same manner as the sets $\{\lambda_k^{DD}\} = \{\lambda_k^{DD}(L)\}$, etc., were defined in (3.4) (using Eq. (3.1)).

THEOREM 4.3. *The set* $\Sigma_1(L; a, b)$ *consists of a collection of* C^1 *curves* $\Gamma_{k,\nu}(L; a, b)$, $k \geqslant 2$, $\nu \in \{\pm\}$, *which have all the properties described in Theorem* 2.1, *except property* (ii). *The vertical asymptotes of the curves* $\Gamma_{k,\nu}(L; a, b)$ *are as in Theorem* 3.1, *with the argument* (L) *of the eigenvalues replaced by* $(L; a)$, *and the horizontal asymptotes are as in Corollary* 3.3, *with the argument* (L) *replaced by* $(L; b)$.

Remark 4.4. It is not clear in this case whether, for any given $k \geqslant 2$, the curves $\Gamma_{k,+}(L; a, b)$, $\Gamma_{k,-}(L; a, b)$ intersect in general. Of course, for certain a, b, the locations of the asymptotes may force the curves to intersect. We also note that the standard Fucik spectrum $\Sigma(L)$ is symmetric with respect to reflection in the diagonal D. This need not be true for the generalized Fucik spectrum $\Sigma(L; a, b)$.

5. HALF-EIGENVALUES OF $(L; a, b)$ AND ASSOCIATED SPECTRAL THEORY

Let $(a, b) \in L^\infty(0, \pi)^2$. We first consider the problem

$$Lu = au^+ - bu^- + \lambda u. \tag{5.1}$$

Let

$$\tilde{\Sigma}_H(L; a, b) = \{(\lambda, u) \in \mathbb{R} \times S_H : (5.1) \text{ holds}\},$$

and let $\Sigma_H(L; a, b)$ be the projection of $\tilde{\Sigma}_H(L; a, b)$ onto $\mathbb{R}$. The problem (5.1) is considered in Section 3 of [2], where elements of the set $\Sigma_H(L; a, b)$ are called *half-eigenvalues* of $(L; a, b)$, and also in [3]. Without loss of generality we may suppose that $a, b > 0$ (which was assumed in Section 4), and also that $\lambda_1 > 0$, since, if not, we may rewrite the problem in the form

$$(L+\tau)\, u = (a+\tau)\, u^+ - (b+\tau)\, u^- + \lambda u,$$

with τ sufficiently large to ensure that $a+\tau, b+\tau, \lambda_1+\tau > 0$. We now have the following result regarding $\tilde{\Sigma}_H(L; a, b)$.

THEOREM 5.1. *For each $k \geqslant 1$, $\nu \in \{\pm\}$, there exists a unique solution*

$$(\lambda_{k,\nu}, \psi_{k,\nu}) \in \mathbb{R} \times S_{k,\nu}$$

of (5.1) $(\lambda_{k,\nu} = \lambda_{k,\nu}(L; a, b),\ \psi_{k,\nu} = \psi_{k,\nu}(L; a, b))$. *If $k' > k \geqslant 1$ then*

$$\lambda_{k',\nu'} > \lambda_{k,\nu}, \qquad \textit{for each} \quad \nu', \nu \in \{\pm\}. \tag{5.2}$$

Proof. The proof is similar to the proof of Lemma 4.1 (here, the continuation argument goes from $\gamma = 0$ to $\gamma = 1$). ∎

Remark 5.2. This theorem is almost the same as Theorem 2 in [2] or Theorem 4 in [3]. The main difference is that the inequality (5.2) is not proved in [2] or [3]. Also, in [2, 3] the functions a, b are constants, and the term λu in (5.1) is $\lambda r u$, where $r \in C^0([0, \pi])$, with $r > 0$. Such a term could easily be included above. Furthermore, estimates on the location of the half-eigenvalues can be obtained from Theorem 1 of [2]. Somewhat similar estimates, in terms of the eigenvalues of L and $a_{\max}, a_{\min}, b_{\max}, b_{\min}$ (in the $L^\infty(0, \pi)$ sense), could be obtained using (4.5) and the continuation construction from $\gamma = 0$ to $\gamma = 1$. A special case of Theorem 5.1 is proved in Theorem 1.1 in [12].

We now prove a monotonicity result for the half-eigenvalues. For $(a_i, b_i) \in L^\infty(0, \pi)^2$, $i = 0, 1$, we write $(a_0, b_0) \leqslant (a_1, b_1)$ if

$$a_0(x) \leqslant a_1(x) \qquad \text{and} \qquad b_0(x) \leqslant b_1(x) \quad \text{a.e. } x \in [0, \pi], \tag{5.3}$$

and we write $(a_0, b_0) < (a_1, b_1)$ if $(a_0, b_0) \leqslant (a_1, b_1)$ and *both* the inequalities in (5.3) hold strictly when x lies in some subset of $[0, \pi]$ having positive measure.

THEOREM 5.3. *If $(a_0, b_0) < (a_1, b_1)$, then for each $k \geqslant 1$, $\nu \in \{\pm\}$,*

$$\lambda_{k,\nu}(L; a_0, b_0) > \lambda_{k,\nu}(L; a_1, b_1).$$

Proof. We again use a continuation argument. Consider the problem

$$Lv = [(1-\gamma)\, a_0 + \gamma a_1]\, v^+ - [(1-\gamma)\, b_0 + \gamma b_1]\, v^- + \mu v,$$

with $\gamma \in [0, 1]$. Consider a fixed k and ν. At $\gamma = i$, $i = 0, 1$, this problem has the unique solution

$$(\mu, v) = (\lambda_{k,\nu}(L; a_i, b_i), \psi_{k,\nu}(L; a_i, b_i)) \in \mathbb{R} \times S_{k,\nu},$$

and the continuation process produces a curve $(\mu(\gamma), v(\gamma)) \in \mathbb{R} \times S_{k,\nu}$, $\gamma \in [0, 1]$, linking these solutions. Also, by calculating $\mu'(\gamma)$ (see (4.5) for the result of a similar calculation) and using the inequalities on a_i, b_i, it can be shown that $\mu'(\gamma) < 0$, $\gamma \in [0, 1]$, which proves the result. ∎

Remark 5.4. A similar proof shows that the monotonicity result also holds for each $k \geqslant 2$ (so that the eigenfunctions change sign) if $(a_0, b_0) \leqslant (a_1, b_1)$ and at least one of the inequalities in (5.3) holds strictly for almost all $x \in [0, \pi]$.

For each $k \geqslant 1$, let $\lambda_{k,\max} = \max\{\lambda_{k,+}, \lambda_{k,-}\}$, $\lambda_{k,\min} = \min\{\lambda_{k,+}, \lambda_{k,-}\}$, and define the open intervals

$$\Lambda_k^0 = \begin{cases} (\lambda_{k,\min}, \lambda_{k,\max}), & \text{if} \quad \lambda_{k,\min} \neq \lambda_{k,\max}, \\ \varnothing & \text{if} \quad \lambda_{k,\min} = \lambda_{k,\max}, \end{cases}$$

$$\Lambda_k^1 = (\lambda_{k,\max}, \lambda_{k+1,\min}), \qquad \Lambda_0^1 = (-\infty, \lambda_{1,\min}).$$

Intuitively, Theorem 5.1 says that the term $au^+ - bu^-$ in Eq. (5.1) "splits apart" the usual eigenvalues λ_k into half-eigenvalues $\lambda_{k,+}, \lambda_{k,-}$ (indeed, the proof carries out this process). The interval Λ_k^0 is the gap between the half-eigenvalues $\lambda_{k,\pm}$ produced by this splitting process, and may be empty if the half-eigenvalues coincide. The inequality (5.2) says that in this splitting process, half-eigenvalues with different values of k do not meet each other, so the interval Λ_k^1 between half-eigenvalues corresponding to k and $k+1$ is non-empty. Also, all these intervals are disjoint and their union comprises the whole of $\mathbb{R}$, except for the half-eigenvalues.

In addition to eigenvalues, linear spectral theory is also concerned with the solvability of inhomogeneous problems. Accordingly, we will consider the solvability of the inhomogeneous equation

$$Lu = au^+ - bu^- + \lambda u - f, \tag{5.4}$$

for general functions $f \in L^2(0, \pi)$, when λ is not a half-eigenvalue (equation (5.4) is the analogue, in the present setting, of equation (3.6) of [5]). Related to equation (5.4), we define the operator $R_\lambda : H \to H$ by

$$R_\lambda(u) = u - L^{-1}(au^+ - bu^- + \lambda u)$$

(since $\lambda_1 > 0$, the operator $L^{-1}: L^2(0, \pi) \to H$ exists and is bounded). Clearly, (5.4) is equivalent to the equation $R_\lambda(u) = -L^{-1}f$. The operator R_λ is positively homogeneous, in the sense that $R_\lambda(tu) = tR_\lambda(u)$ for any $t \geq 0$ and $u \in H$. Also, since the mapping $u \to au^+ - bu^- + \lambda u : H \to L^2(0, \pi)$ is compact, the operator $I - R_\lambda$ is compact and so, letting $B_r(c)$ denote the ball in H with centre c and radius r, the Leray-Schauder degree, $\deg(R_\lambda, B_1(0), 0)$, is well-defined for all $\lambda \notin \Sigma_H(L; a, b)$, see [7], and $\deg(R_\lambda, B_1(0), 0)$ is constant on any interval Λ_k^0, Λ_k^1.

THEOREM 5.5. (A) *If* $\lambda \in \Lambda_k^1$, *for some* $k \geq 0$, *then*:

(i) $\deg(R_\lambda, B_1(0), 0) = (-1)^k$;

(ii) *for any* $f \in L^2(0, \pi)$, *Eq.* (5.4) *has a solution* $u \in H$.

(B) *If* $\lambda \in \Lambda_k^0$, *for some* $k \geq 1$, *then*:

(i) $\deg(R_\lambda, B_1(0), 0) = 0$;

(ii) *there exists* $f \in L^2(0, \pi)$ *such that equation* (5.4) *has no solution*;

(iii) *there exists* $z_\lambda \in L^2(0, \pi)$ *with the following property*: *for any* $h \in L^2(0, \pi)$ *there is a number* $T(h)$ *such that, if* $t > T(h)$ *and* $f = tz_\lambda + h$, *then* (5.4) *has at least* 2 *solutions.*

Proof. Suppose that $\lambda \in \Lambda_k^1$ for some $k \geq 1$. We again use the constructions in the proof of Lemma 4.1. Since the functions $\mu_{k, \pm}$, $\mu_{k+1, \pm}$, constructed there are C^1, we can choose a point $\lambda_0 \in (\lambda_k, \lambda_{k+1})$ and a C^1 function $\mu: [0, 1] \to \mathbb{R}$ such that $\mu(0) = \lambda_0$, $\mu(1) = \lambda$, and

$$\mu_{k, \pm}(\gamma) < \mu(\gamma) < \mu_{k+1, \pm}(\gamma), \qquad \gamma \in [0, 1]$$

(see the final part of the proof of Lemma 4.1). A simple modification of this argument also proves this result for the case $k = 0$ (simply by omitting any reference to $\mu_{k, \pm}$ in this case). Now, let $T_\gamma(u) = u - L^{-1}(\gamma(au^+ - bu^-) + \mu(\gamma)\, u)$, $u \in H$, $\gamma \in [0, 1]$. By construction, for each $\gamma \in [0, 1]$ there is no non-trivial solution of the equation $T_\gamma(u) = 0$. Also, $T_0(u) = u - \lambda_0 L^{-1}u$, $T_1(u) = R_\lambda(u)$, and from Theorem 8.10 of [7],

$$\deg(T_0, B_1(0), 0) = (-1)^k.$$

Hence part (A)-(i) follows from standard continuity properties of the degree. Now, part (A)-(ii) follows from part (A)-(i), the positive homogeneity of the operator R_λ, and standard properties of the degree, see parts (D4) and (D5) of Theorem 8.2 in [7].

To prove part (B) of the theorem we first prove part (B)-(ii)—by the previous argument, part (B)-(i) will then follow immediately. The proof of part (B)-(ii) is based on the proof of part (i) of Proposition 1 of [5], which proves a similar result in the case of Dirichlet boundary conditions. The different boundary conditions here makes the proof somewhat longer. If either boundary condition is Dirichlet then the proof in [5] can be used, so we assume that neither is Dirichlet. Then we may rewrite condition (1.3) in the form $u'(\pi)-du(\pi)=0$, for some $d\in\mathbb{R}$. We will, essentially, follow Dancer's notation and constructions, merely noting the necessary changes, and we refer to [5] for more details.

Let $\Phi_\pm$ denote the solutions of (1.1) with $\Phi_\pm(0)=\pm 1$ and satisfying (1.2), and let

$$B(\lambda)=(\Phi'_+(\pi)-d\Phi_+(\pi))(\Phi'_-(\pi)-d\Phi_-(\pi)).$$

Now, $\lambda\in\Sigma_H(L;a,b)\Leftrightarrow B(\lambda)=0$, and the sign of $B(\lambda)$ changes as λ crosses a half-eigenvalue. We will show that part (B)-(ii) holds when $B(\lambda)>0$, so by part (A)-(ii),

$$\bigcup_{k\geqslant 1}\Lambda_k^0=\{\lambda\colon B(\lambda)>0\},\qquad \bigcup_{k\geqslant 0}\Lambda_k^1=\{\lambda\colon B(\lambda)<0\}.$$

Only the case $\Phi'_\pm(\pi)-d\Phi_\pm(\pi)>0$ will be considered (the other case is similar). We first construct f. Choose $x_0\in(0,\pi)$ such that $\Phi_+(x)\,\Phi_-(x)\neq 0$ for $x\in[x_0,\pi)$. Now consider the problem

$$-(pv')'+qv=av+\lambda v-1,\qquad \text{in } (x_l,\pi), \tag{5.5}$$

$$v(x_l)=0,\qquad v'(x_l)\geqslant 0, \tag{5.6}$$

$$v'(\pi)=dv(\pi), \tag{5.7}$$

for arbitrary $x_l\in(0,\pi)$. Using Prüfer angle methods (see [4, Chap. 8]), it can be shown that there exists a $\delta>0$ sufficiently small so that if $x_l\in[\pi-\delta,\pi)$ then any solution v of the initial value problem (5.5), (5.6), has no zero in $(x_l,\pi]$, and the complete boundary value problem (5.5)–(5.7) has no solution. Suppose also that $\pi-\delta>x_0$, and define f to be 1 on $[\pi-\delta,\pi)$ and 0 on $(0,\pi-\delta)$.

With this f, let Φ_α denote the solution of the differential equation corresponding to (5.4) with $\Phi_\alpha(0)=\alpha$ and satisfying (1.2). Hence, Φ_α is a solution of (5.4) if and only if it satisfies (5.7) (i.e., (1.3)). Furthermore, any solution of (5.4) must have the form of Φ_α, for some $\alpha\in\mathbb{R}$. We will show that Φ_α cannot satisfy (5.7) for any $\alpha\in\mathbb{R}$, which will prove the result.

We first consider the case $\alpha<0$, and we note that, since $f\equiv 0$ on $(0,\pi-\delta)$, the function Φ_α is a scalar multiple of Φ_- on this interval. Now suppose that $\Phi_{-}(x) > 0$ for $x \in [x_0, \pi)$ (we will consider the other case below). Following Dancer's method of proof we find that $\Phi_\alpha(x) > 0$, $x \in (x_0, \pi)$, and $W(\pi) < 0$ (where $W = \Phi_\alpha \Phi'_{-} - \Phi'_\alpha \Phi_{-}$). Thus, if Φ_α satisfies (5.7) then

$$0 > W(\pi) = \Phi_\alpha(\pi)(\Phi'_{-}(\pi) - d\Phi_{-}(\pi)) \geqslant 0,$$

which is a contradiction.

Next, suppose that $\Phi_{-}(x) < 0$ for $x \in [x_0, \pi)$, and hence $\Phi_\alpha(x) < 0$ for $x \in [x_0, \pi - \delta]$. If $\Phi_\alpha(x) < 0$ for all $x \in [x_0, \pi)$, then calculations similar to those for the previous case show that Φ_α cannot satisfy (5.7) (in this case, $W(\pi) > 0$). Now suppose that Φ_α has a zero in $(\pi - \delta, \pi)$ and let x_1 be the smallest such zero. Clearly, $\Phi'_\alpha(x_1) \geqslant 0$, so if Φ_α satisfies (5.7) then it is a solution of the boundary value problem (5.5)–(5.7), with $x_l = x_1$, which is again a contradiction.

We conclude that Φ_α cannot be a solution of (5.4) when $\alpha < 0$. Similar arguments (using Φ_{+}) show that Φ_α cannot be a solution when $\alpha > 0$. When $\alpha = 0$, we have $\Phi_0 \equiv 0$ on $(0, \pi - \delta)$, so if Φ_0 satisfied (5.7) then Φ_0 would be a solution of (5.5)–(5.7), with $x_l = \pi - \delta$, which is again a contradiction. This completes the proof of part (B)-(ii) of the theorem.

To prove part (B)-(iii) we choose a function $\phi_\lambda \in H$ such that $D_u R_\lambda(\phi_\lambda)$ is non-singular, and let $z_\lambda = -LR_\lambda(\phi_\lambda)$ (we will show below that such a ϕ_λ exists). Then ϕ_λ is an isolated solution of the equation $R_\lambda(u) = -L^{-1} z_\lambda$, with index ± 1. Hence, by continuity properties of the degree and part (B)-(i), there exists sufficiently small numbers $r_1, r_2, r_3 > 0$ such that if $\|h\| < r_3$, then

$$\deg(R_\lambda, B_1(0), -r_1 L^{-1}(z_\lambda + h)) = 0,$$
$$\deg(R_\lambda, B_{r_2}(r_1 \phi_\lambda), -r_1 L^{-1}(z_\lambda + h)) \neq 0,$$

and so there exist solutions of the equation $R_\lambda(u) = -r_1 L^{-1}(z_\lambda + h)$ in $B_{r_2}(r_1 \phi_\lambda)$ and in $B_1(0) \backslash B_{r_2}(r_1 \phi_\lambda)$ (we assume that the numbers r_i are sufficiently small that $B_{r_2}(r_1 \phi_\lambda) \subset B_1(0)$). The result as stated in the theorem now follows by scaling and using the positive homogeneity of R_λ. This proves part (B)-(iii) if we can show that suitable ϕ_λ exists. To do this we first choose $\phi_\lambda \in H$ such that ϕ_λ has only simple zeros and $\phi_\lambda > 0$ in $(0, \pi)$. Then,

$$D_u R_\lambda(\phi_\lambda)\, v = L^{-1}(L - a - \lambda)\, v, \qquad v \in H,$$

and hence $D_u R_\lambda(\phi_\lambda)$ is non-singular for all λ which are not eigenvalues of the Sturm–Liouville operator $L - a$, so this ϕ_λ will suffice for such λ. Now suppose that λ is such an eigenvalue, and suppose that there exists an open interval $\tilde{J}$ on which $a > b$ or $a < b$ (since $\lambda \in \Lambda_k^0$ we must have $a \neq b$ in $L^\infty(0, \pi)$). Choose $\phi_\lambda \in H$ so that ϕ_λ has only simple zeros and

$\phi < 0$ in an open subinterval $J \subset \tilde{J}$ and $\phi > 0$ in $(0, \pi) \backslash \bar{J}$. Then $D_u R_\lambda(\phi_\lambda)$ is non-singular if the Sturm–Liouville operator $L - a\chi_{(0, \pi)\backslash \bar{J}} - b\chi_J - \lambda$ is non-singular, and this is true for any sufficiently small interval J by standard monotonicity properties of eigenvalues. For general $a \neq b$ in $L^\infty(0, \pi)$ a similar argument works, using rather more measure theory—for brevity we will omit the details. This completes the proof of the theorem. ∎

Remark 5.6. The above results can be regarded as a generalization, to the half-linear case, of certain results from the usual linear spectral theory of Sturm–Liouville problems. When $a = b$ the problem reduces to the linear case—so all the half eigenvalues coincide and the intervals Λ_k^0 are empty. In the linear case, the degree $\deg(R_\lambda, B_1(0), 0)$ changes by 2 as λ crosses a simple eigenvalue, which in the above context can be heuristically regarded as crossing two coincident half-eigenvalues, each of which contributes a change of 1. The second part of Theorem 5.5 has no analogue in this case.

6. NON-LINEAR PROBLEMS

In this section we consider nonlinear problems of the form

$$Lu = G(u) - f, \tag{6.1}$$

where $f \in L^2(0, \pi)$ and $G\colon H \to L^2(0, \pi)$ is a Nemitski operator of the form $G(u)(x) = g(x, u(x))$, $x \in [0, \pi]$ (see Section 1.2 of [1]), and the function g has the form

$$g(x, \xi) = a(x)\, \xi^+ - b(x)\, \xi^- + \tilde{g}(x, \xi),$$

where $a, b \in L^\infty(0, \pi)$, and $\tilde{g}$ is a Carathéodory function on $[0, \pi] \times \mathbb{R}$ satisfying $|\tilde{g}(x, \xi)| \leqslant A(x) + B(|\xi|)|\xi|$, $(x, \xi) \in [0, \pi] \times \mathbb{R}$, with $A \in L^2(0, \pi)$ and $\lim_{|\xi| \to \infty} B(|\xi|) = 0$.

THEOREM 6.1. *If* $0 \in \Lambda_k^1$, *for some* $k \geqslant 0$, *then for any* $f \in L^2(0, \pi)$ *equation* (6.1) *has a solution* $u \in H$. *If* $0 \in \Lambda_k^0$, *for some* $k \geqslant 1$, *then*:

(i) *there exists* $f \in L^2(0, \pi)$ *such that equation* (6.1) *has no solution*;

(ii) *there exists* $z \in L^2(0, \pi)$ *with the following property*: *for any* $h \in L^2(0, \pi)$ *there is a number* $T(h)$ *such that, if* $t > T(h)$ *and* $f = tz + h$, *then* (6.1) *has at least* 2 *solutions.*

Proof. The proof of the first two results is similar to the proof of parts (iii) and (iv) of Theorem 5 in [5], using the results of Theorem 5.5 (in the present setting, the analogue of Lemma 2 in [5] for the operator G follows from the results in Section 1.2 of [1]). The proof of the final result is similar to the proof of part (b) of Theorem 1.4 in [12]. ∎

Special cases of Theorem 6.1 are proved in Theorem 5 of [5] and Theorem 1.4 of [12].

Of course, it may not be easy to check the hypotheses $0 \in \Lambda_k^0$ or $0 \in \Lambda_k^1$ in Theorem 6.1. However, we can give a sufficient condition for these hypotheses to hold in terms of the location of the set of values $\{(a(x), b(x)) : x \in [0, \pi]\}$, relative to the (standard) Fucik spectrum $\Sigma(L)$ of L.

Lemma 6.2. *Suppose that $(\alpha_0, \beta_0) < (a, b) < (\alpha_1, \beta_1)$ for some $(\alpha_i, \beta_i) \in \mathbb{R}^2$, $i = 0, 1$, (here, we regard (α_i, β_i) as constant functions on $[0, \pi]$). Suppose also that $k \geqslant 1$. Then:*

(i) *if $(\alpha_i, \beta_i) \in \Gamma_{k+i}$, $i = 0, 1$, and the rectangle $R = (\alpha_0, \alpha_1) \times (\beta_0, \beta_1)$ does not intersect $\Sigma(L)$, then $0 \in \Lambda_k^1$;*

(ii) *if $(\alpha_1, \beta_1) \leqslant (\lambda_1, \lambda_1) \in \Gamma_1$, then $0 \in \Lambda_0^1$ (the condition $(\alpha_0, \beta_0) < (a, b)$ is unnecessary here);*

(iii) *if $(\alpha_i, \beta_i) \in \Gamma_k$, $i = 0, 1$, then $0 \in \Lambda_k^0$.*

Proof. It is clear from the definitions that if $(\alpha, \beta) \in \mathbb{R}^2$ lies on the curve $\Gamma_{k,v}$, for some k, v, then $\lambda_{k,v}(L; \alpha, \beta) = 0$ (regarding (α, β) as constant functions). Hence the lemma follows immediately from Theorem 5.3. ∎

Slightly different sufficient conditions, along the lines of Lemma 6.2, could also be obtained using Remark 5.4.

The intersection condition in part (i) of Lemma 6.2 is to rule out the open rectangle R intersecting any of the curves $\Gamma_{k,\pm}$, $\Gamma_{k+1,\pm}$—from the geometry of the curves in $\Sigma(L)$ and the assumption that $(\alpha_i, \beta_i) \in \Gamma_{k+i}$, no other intersection of R and $\Sigma(L)$ is possible. In part (iii) of the lemma, the inequalities on (α_i, β_i) and the geometry of the curves $\Gamma_{k,\pm}$ imply that the points (α_i, β_i) must lie on different curves $\Gamma_{k,\pm}$, and the rectangle R lies between these curves.

Theorem 6.1, together with Lemma 6.2, says that if the set of values $\{(a(x), b(x)) : x \in [0, \pi]\}$ lies in a rectangle situated between adjacent curves $\Gamma_{k,v}$, $\Gamma_{k+1,v'}$ of the Fucik spectrum of L, then Eq. (6.1) is solvable for all $f \in L^2(0, \pi)$. Many theorems of this sort have been proved, see for instance, Theorem 1 in [10], Theorem 1 in [8], and the references therein. The results in these papers are rather more general than above in that, instead of assuming the existence of the limits $\lim_{\xi \to \pm\infty} g(x, \xi)/\xi$, the ranges of the functions $\limsup_{\xi \to \pm\infty} g(x, \xi)/\xi$ and $\liminf_{\xi \to \pm\infty} g(x, \xi)/\xi$ are assumed to lie in appropriate rectangles as in parts (i) and (ii) of Lemma 6.2. Similar results could be proved using the above results but for brevity we will not consider these further. The second situation in Theorem 6.1 has not been considered so often.

The Fucik spectrum was introduced to deal with equations of the form (6.1), when the functions a, b, are constants α, β; see [5, 9]. The solvability of equation (6.1) can then be related directly to the location of the point (α, β) relative to the Fucik spectrum. When the limits a, b are not constant this relationship is rather more problematic, and is usually expressed in terms of sufficient conditions similar to those in parts (i) and (ii) of Lemma 6.2. Looking at the problem in terms of the spectrum $\Sigma_H(L; a, b)$ seems to give rather more conceptually natural necessary and sufficient conditions for both cases in Theorem 6.1 (although these conditions may not be easy to verify).

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